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6 DIAGONALIZATION BY GROUP MATRICES.

10 Donald E. Maurer

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Diagonalization by Group Matrices

In a 1955 paper, O. Taussky [6] studied rational integral matrices of the form A^tA (A^t denotes the transpose of A) where A is an integral unimodular circulant. Subsequently this work was extended [1] to arbitrary integral circulants of prime dimension. More generally, let R be a commutative ring with an identity and let G be a finite group. A G - group matrix A (over R) is a matrix of the form $A = \sum_{g\varepsilon G} a_g P(g)$ where $a_g \varepsilon R$ and P is the left regular representation. Recently Thompson and Garbanati [3] studied the problem of deciding when two non-singular group matrices S and T are G - congruent; i.e. when is there an invertable group matrix A such that $S = A^tTA$ over R. Garbanati [2] obtained computable criteria when G is abelian and R is a field. The purpose of this paper is to study the related problem: given a group G; when is a matrix G - congruent to a diagonal matrix.

Although this paper in some respects generalizes the results of earlier work, its interest lies also in its connection to two problems of classical interest. First, it is evident that any matrix congruent to a diagonal matrix must be symmetric and so represents a quadratic form. It is well known that over a field of characteristic not two a quadratic form can be diagonalized. We ask here when it can be diagonalized by a transformation whose matrix is a group matrix. The second connection is far less apparent. A fundamental problem in the theory of algebraic numbers is the construction of

an extension $K|F$ of fields having a prescribed Galois group G; when the group G is abelian, this is the central problem of class-field theory. Hasse [4] found that it was possible to give a description of abelian extensions in terms of factor systems or 2-cocycles of G. More generally, certain 2-cocycles define an associative, commutative and semi-simple algebra over F, called a Galois algebra [4] on which G acts as a group of F - automorphisms. We show that when G is abelian and $R = F$, a matrix S which is G-congruent to a diagonal matrix determines such a 2-cocycle. A A factorization $S = A^t DA$, where A is a group matrix and D diagonal, is then determined from the structure of the corresponding Galois algebra.

In the first section we give a strong necessary condition for a matrix S to be congruent to a diagonal; the condition holds without restriction on G or R. It is expressed in terms of two invariants Θ_S and Φ_S; the first is the sum of all entries in S, while the second is a certain polynomial determined by S and the group G. When Θ_S is non-zero, a diagonal matrix congruent to S must have on its diagonal a scalar multiple of the zeros of Φ_S arranged in some order. Therefore, up to a scalar factor, there are only finitely many possible diagonal matrices congruent to S. However, when Θ_S is zero we will show that if S is congruent to a diagonal matrix then Φ_S is identically zero. The problem appears to be more difficult, in this case and, apart from some necessary conditions, and an example in section three, will be ignored.

The main results of the paper are given in section two. In this section we assume G is abelian and $R = F$ is a field. Then the invariant, in addition to Φ_S, which determines whether or not S is congruent to a diagonal matrix is a certain function p_S from $G \times G$ to the field of the n^{th} roots of unity over F (where $n = |G|$); in particular $p_S(1, 1) = \theta_S$. When we restrict S so that p_S is always non-zero we obtain, in addition to the connection with Galois algebras, a unique diagonalization: any two factorizations $S = A_1^t D_1 A_1 = A_2^t D_2 A_2$ (where A_i is a non-singular group matrix and D_i is diagonal) are <u>equivalent</u> in the following sense. There is a unit $u \varepsilon R$ and a permutation group matrix W such that $A_2 = u^{-1}(W^t A,)$ and $D_2 = W^t D_1 W$. When n is a prime, all nxn non-group matrices which are congruent to a diagonal satisfy this restriction on p. On the other hand, whenever S is a group matrix, $p_S(g, h) = 0$ unless $h = g^{-1}$. There may be a number of inequivalent factorizations in this case and actual computations can be quite difficult. Another class of matrices for which we are are able to obtain computable criteria are those of dimension 2^m when G is an elementary abelian 2 - group of order $|G| = 2^m$.

In section three the quadratic case $n = 2$ is presented as an example illustrating the general theory. The differences between non-zero θ_S and $\theta_S = 0$ are clearly evident in this simple case.

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§1 Preliminary Results

It is the purpose of this section to establish notation and introduce some of the invariants which play a role in diagonalization by group matrices. Later it will be necessary to impose certain restrictions on both R and G, however the results of this section are true in general.

Let $n = |G|$ and enumerate the elements of G according to a fixed order $1 = g_1, g_2, \ldots, g_n$. Then any $n \times n$ R - matrix $S = (s_{ij})$ is uniquely associated with a function $s\colon G \times G \to R$ defined by $s(g_i, g_j) = s_{ij}$. Using this notation it is convienent to write $S = (s(g, h))$. Similarly, a diagonal matrix D can be expressed in the form $D = (d(g))$, where $d\colon G \to R$ is an appropriate function. Then S is a G-group matrix if $s(g,h) = s(1, g^{-1}h)$ for all $g, h \in G$. Now, for any matrix S let $S^G = (s^G(g,h))$ be defined by

$$s^G(g,h) = \sum_{u \in G} s(ug,uh) \quad ; \text{ for } g,h \in G .$$

Evidently $s^G(g, h) = s^G(1, g^{-1}h)$ and so S^G is a group matrix. A straightforward calculation shows that if A is a group matrix then $(SA)^G = S^G A$, for if we let $C = SA$ then

$$c^G(g,h) = \sum_{u,v \in G} s(ug,uv)a(uv,uh)$$

but $a(uv, uh) = a(1,v^{-1}h)$ is independent of u, whence

$$c^G(g,h) = \sum_{u\in G} a(u,h)\; S^G(g,u) \; .$$

In particular, if S is a group matrix then $S^G = nS$.

We now introduce the polynomial $\Phi_S \in R[x]$ defined by

$$\Phi_S(x) = \det\,(S - xS^G),$$

and proceed to describe some of its properties. First, it is easy to see that the coefficient of x^n is $(-1)^n \det S^G$; however the following stronger result holds.

Lemma 1. <u>If</u> S^G <u>is</u> <u>invertable</u> <u>then</u>

$$(-1)^n\, \Phi_S\,(x) = (x^n - x^{n-1}) \det S^G + g(x),$$

<u>where</u> <u>deg</u>$(g(x)) < n-1$.

proof. Let $B = S(S^G)^{-1}$ so that

$$\Phi_S(x) = \det S^G \det\,(B - xI_n),$$

where I_n is the nxn identity. Now it is only necessary to show that trace$(B) = 1$. But $BS^G = S$ and therefore $B^G\, S^G = S^G$, whence $B^G = I_n$. However, the diagonal elements of B^G are all equal to trace(B).

The following properties are immediate.

Corollary 1. The *polynomial* Φ_S *satisfies*:

(i) *deg* $\Phi_S = n$ *if and only if* *det* $S^G \neq 0$.

(ii) *If* $S^G = 0$ *then* $\Phi_S(x) = \underline{\det} S$.

(iii) $\Phi_S(0) = \underline{\det}\, S$.

(iv) *If* S^G *is invertable, the zeros of* Φ_S *sum to one*.

(v) *If* S *is a group matrix then* $\Phi_S(x) = \Phi_S(0).(1-nx)^n$.

proof. If S is a group matrix then $S^G = nS$, and so

$\Phi_S(x) = \det S.(1-nx)^n$; using (iii) we obtain (v).

We remark that if S is a singular group matrix the above shows that $\Phi_S(x) \equiv 0$.

Our next lemma will indicate the significance of Φ_S; but first we must introduce some terminology. In what follows A denotes an invertable G-group matrix and D denotes a diagonal matrix. A pair (A,D) will be called a *factorization* of S if $S=A^tDA$. A factorization is, at most, unique up to equivalence as defined in the introduction.

Lemma 2. *Let* (A,D) *be a factorization of* S. *Then*

$$\Phi_S(x) = (\underline{\det}\, A)^2 \prod_{g\varepsilon G} (d(g) - \theta x),$$

where $\theta = \sum_{g\varepsilon G} d(g)$ *is the sum of the entries of* D.

proof. Writing $A = (a(g,h))$ we have

$$S^G(g,h) = \sum_{u,v\varepsilon G} a(1, v^{-1}ug)\, d(v) a(1, v^{-1}uh)$$

$$= \sum_{v\varepsilon G} d(v) \sum_{\omega\varepsilon G} a(1,\omega^{-1}g)\, a\,(1,\omega^{-1}h).$$

That is

(1) $$S^G = \theta\, A^t A.$$

The lemma then follows from the definition of Φ_S.

It is now easy to prove a necessary condition for S to have a factorization. We denote by Θ_S the sum of the entries of S.

Proposition 1. <u>If a matrix</u> S <u>is</u> G - <u>congruent to a diagonal matrix then</u>

(i) $\Theta_S = 0 <=> S^G = 0$

$<=> \underline{\det}\, S^G = 0$.

(ii) <u>If</u> $\Theta_S \neq 0$ <u>then</u> $\Phi_S(x)$ <u>factors into</u> n <u>linear factors in</u> $R\,[x]$. <u>Moreover, if</u> R <u>is an integral domain with quotient field</u> F, <u>the zeros of</u> Φ_S <u>lie in</u> F, <u>and if</u> (A,D) <u>is a factorization of</u> S <u>then</u> $d(g)/\theta$ <u>is a zero of</u> Φ_S <u>for each</u> $g\varepsilon G$.

Proof. We have

$$\Theta_S = \sum_{g \varepsilon G} s^G(1,g)$$

$$= \theta \sum_{g,h \varepsilon G} a(h, 1)a(h,g)$$

$$= \theta \left(\sum_{h \varepsilon G} a(1,g) \right)^2$$

This together with equation (1) shows that $\Theta_S = 0$ implies $S^G = 0$ and conversely det $S^G = 0$ implies $S^G = 0$. This proves (i). Condition (ii) is simply a restatement of lemma 2.

As an example let $R = Z$ be the ring of rational integers. If $S = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$ then (since G must be cyclic of order 2) S^G is not zero but $\Theta_S = 0$, so S cannot be factored. Similary if $S = \begin{pmatrix} 1 & -3 \\ -3 & 0 \end{pmatrix}$ then

$$\Phi_S(x) = 35\ (x-x^2) - 9\ ,$$

and this polynomial is irreducible over the rational field. Again S cannot be congruent to a diagonal matrix.

It is clear from the proposition that if Θ_S is non-zero we are able to obtain a great deal of information about the diagonal matrix in any factorization of S; but when Θ_S is zero and S has a factorization, then Φ_S is identically zero. Hence our

results in the non-zero case are more satisfactory. If we assume Θ_S is not zero and restrict R to an integral domain with quotient field F, according to the proposition each factorization (A,D) defines a bijection ν of G onto the set of zeros of Φ_S which is characterized by the relations $d(g) = \theta\nu(g)$ for all $g \in G$. Thus D is completely determined by θ and ν. Note that if (A', D') is an equivalent factorization, then there is some $e \in G$ such that $\nu'(g) = \nu(eg)$ for all $g \in G$.

Before proceeding to the next section, we remark that if F is an algebraic number field, there is an algorithim [7] for factoring polynomials in $F[x]$. Thus we can determine if (ii) of proposition one is satisfied.

§2 The Abelian Case

Throughout most of this section we will take $R = F$, and suppose that the characteristic of F does not divide 2n. Also, G will be an Abelian group. We begin this section with a brief account of material in [2], [4] and [5] which will be necessary for our analysis.

Let $\Delta = n^{-\frac{1}{2}}(\chi(g))$ denote the matrix whose (i, j)-th position contains the element $n^{-\frac{1}{2}}\chi_i(g_j)$, where $\chi_1, \ldots, \chi_n$ are the characters of G ordered so that the map $\chi_i \to g_i$ is an isomorsphism of the character group $\hat{G}$ onto G. If A is a group matrix, define for each $\chi \in \hat{G}$

$$\omega_\chi(A) = \sum_{g \in G} a(1,g)\,\chi(g).$$

Then

(2) $$\Delta A \Delta^{-1} = \text{diag}\,(\omega_{\chi_1}, \ldots, \omega_{\chi_n}).$$

In a more compact notation, the right hand side can be written diag (ω_χ). Now let F_d be the field obtained from F by adjoining the d-th roots of unity, and denote by $G(F_d|F)$ the Galois group of F_d over F. Now if $d=d_\chi$ (the order of χ), then $\chi(g) \varepsilon F_d$ for all $g \varepsilon G$. Moreover, for each $\pi \varepsilon G(F_n|F)$ the character $\pi o \chi$ defined by $(\pi \circ \chi)(g)=\pi(\chi(g))$ is also of order d. For A the relations $\pi(\omega_\chi) = \omega_{\pi o \chi}$ are evident. Conversely, suppose we have chosen elements $\{\omega_\chi\}_{\chi \varepsilon G}$ such that $\omega_\chi \varepsilon F_d$ $(d = d\chi)$, and $\pi(\omega_\chi) = \omega_{\pi o \chi}$ for all $\pi \varepsilon G(F_n|F)$. Then the n relations

(3) $$a(1,g) = n^{-1} \sum_\chi \chi(g^{-1})\, \omega_\chi$$

define a group matrix $A = (a(g, h))$ with $\omega_\chi = \omega_\chi(A)$.

The non-zero elements of F_n, denoted by $\dot{F}_n$, can be regarded as a trivial $\hat{G}$ - module. Suppose that u is a 2-cocycle of $\hat{G}$ with coefficients in F_n satisfying:

(i) $u(\chi, \psi) = u(\psi, \chi)$ all $\chi, \psi \varepsilon \hat{G}$

(ii) $u(1,1) = 1$

(iii) $\pi\,(u(\chi, \psi)) = u(\pi o \chi, \pi o \psi)$ for $\pi \varepsilon G(F_n|F)$.

Let Λ_n be an n-dimensional vector space over F_n with basis $\{w_\chi\}_{\chi\varepsilon\hat{G}}$. Then Λ_n can be made into an associative, commutative and semi-simple F_n- algebra by defining multiplication according to the relations

$$w_\chi \, w_\psi = u(\chi,\psi) w_{\chi\psi} \; .$$

Moreover, G operates on Λ_n as a group of F_n- automorphisms according to the rule $g(w_\chi) = \chi(g^{-1}) \, w_\chi$ for all $g \varepsilon G$. The elements

$$w_g = n^{-1} \sum_\chi \chi(g^{-1}) \, w_\chi \quad \text{(each } g\varepsilon \text{ G.)}$$

form a normal basis for Λ_n. Now, for each $\pi \,\varepsilon\, G\,(F_n|F)$ we define a bijective F- linear map $\phi_\pi: \Lambda_n \to \Lambda_n$ by

$$\phi_\pi\left(\sum a_\chi w_\chi\right) = \sum \pi(a_\chi) w_{\pi\circ\chi} \; .$$

The map ϕ_π is well-defined, and since $\phi_\pi(w_\chi w_\psi) = \phi_\pi(w_\chi) \cdot \phi_\pi(w_\psi)$ it is a ring automorphism. Let

$$\Lambda = \{x\varepsilon\Lambda_n : \phi_\pi(x) = x \text{ for all } \pi \,\varepsilon\, G(F_n|\, F)\}$$

Since

$$\phi_\pi(w_g) = n^{-1} \sum_\chi \pi\circ\chi(g^{-1}) \, w_{\pi\circ\chi} = w_g \; ,$$

it is easy to see that Λ is an F-algebra with basis $\{w_g\}_{g\varepsilon G}$ on which G acts as a group of F-automorphisms via the rule $g(w_h) = w_{gh}$ for g, hεG. Such an algebra is called a Galois algebra [4]. Conversely, any Galois algebra will determine a 2-cocycle of $\hat{G}$ with coefficients in $\dot{F}_n$.

Now define structure constants $\{C^k_{g,h}\}$ for g, h,kεG by the equations

$$w_g \cdot w_h = \sum_{k \in G} C^k_{g,h} w_k, \qquad C^k_{g,h} \in F.$$

Since $\{w_g\}$ is a normal basis, we have $C^k_{g,h} = C^1_{k^{-1}g, k^{-1}h}$; so the structure constants are determined by the matrix C, where $c(g,h) = C^1_{g,h}$. We shall call C a structure matrix for Λ.

Lemma 3. The coefficients $c(g,h)$ are given by

$$w(g,h) = n^{-2}\sum_{\chi,\Psi} \chi(g^{-1})\Psi(h^{-1})u(\chi,\Psi).$$

Proof: Let g, hεG be fixed and set $a_k = C^k_{g,h}$. Then

$$w_g w_h = n^{-1}\sum_{\chi}\left(\sum_k \chi^{-1}(k)a_k\right) w_\chi$$

On the other hand, expressing w_g and w_h directly in terms of $\{w_\chi\}$ and equating corresponding coefficients gives, for all $\Psi\varepsilon\hat{G}$

$$\sum_k \Psi(k)a_k^{-1} = \frac{\Psi(h^{-1})}{n}\sum_\chi \chi(g^{-1}h)u(\chi,\chi^{-1}\Psi);$$

whence, solving for a_k and setting $k = 1$, we obtain the required expression for $\Phi(g,h)$.

Now applying the Wedderburn theory, we find that Λ is isomorphic as a Galois algebra to a direct sum of isomorphic fields; that is, there is a field extension $K|F$ such that

$$\Lambda = \bigoplus_{i=1}^{m} K, \text{ for some } m.$$

It can be shown [5] that K is the splitting field of Φ_C moreover, there is an ordering $\{\eta_g\}_{g\varepsilon G}$ of the zeros of Φ_C that the Galois algebra $\Lambda \otimes_F K$ over K has a normal basis consisting of idempotents $\{e_g\}_{g\varepsilon G}$ given by

$$(4) \qquad e_g = \sum_h \eta_{hg} w_{h^{-1}}$$

We are now in a position to study the factorizations (A,D) for a given F-matrix S. The following more general formulation will bring into full interplay the link between Galois algebra and factorizations. We wish to determine if there exists an extension field $K|F$ such that S has a factorization (A,D) over K. First, consider a reduction. Obviously we need only determine all inequivalent factorizations; moreover in the proof of proposition one we showed that if (A,D) is a factorization of S then

$$\sum_g d(g) = \Theta_S \pmod{K^2},$$

whence it is evident that S must have a factorization satisfying $\Theta_S = \sum d(g)$. For such a factorization we have $d(g) = \Theta_S \nu(g)$, so that if Θ_S is non-zero, D is completely determined by Φ_s and the bijection ν. We will mainly be interested in factorizations which satisfy this condition.

Now let S be an arbitrary nxn F-matrix such that Θ_S is non-zero, let ν denote a bijection of G to the set of zeros of $\Phi_S(x)$; and set

$$q_S(\nu;\chi) = \sum_g \chi(g^{-1})\nu(g)$$

for each $\chi \in G$ and

$$p_S(\chi,\Psi) = \sum_{g,h} \chi(g^{-1})\Psi(h^{-1})s(g,h)$$

for each pair $\chi,\Psi \in \hat{G}$.

We see that corollary one implies that $q_S(\nu;1) = 1$ independently of ν. Also note that $p_S(1,1) = \Theta_S$. The polynomial $\Phi_S(x)$ and the map $p_S : \hat{G} \times \hat{G} \to F_n$ are the fundamental invariants which determine K and (A,D). In order to show this suppose that

(5) $\quad S = A^t DA \quad (\text{with } \Theta_S = \sum d(g))$

is a factorization over K.

Replacing S by $\Delta S\Delta^{-1}$ in equation (5) and setting $P = \Delta S\Delta^{-1}$ and $Q = \Delta D\Delta^{-1}$, we obtain the matrix equation

$$P \cdot \operatorname{diag}(\omega_\chi^{-1}) = \operatorname{diag}(\omega_{\chi^{-1}}) \cdot Q ,$$

where $\omega_\chi = \omega_\chi(A)$. This follows directly from the relation (2) for group matrices, and the fact that $\omega_\chi(A^t) = \omega_{\chi^{-1}}(A)$. Equate

the entries term by term in the above equation and replace χ^{-1} by χ to obtain the following equivalent system of quadratic equations

$$\text{(6)} \qquad \omega_\chi \, \omega_\Psi q_S(\nu; \chi\Psi) = p_S(\chi,\Psi)\Theta_S^{-1} \qquad (\text{all } \chi,\Psi \in \hat{G}).$$

We will say that a solution of (6) $\{\omega_\chi\}_{\chi\varepsilon\hat{G}}$ is <u>compatible with</u> K if $\omega_\chi \varepsilon K_n$ all χ, and for each $\Pi \varepsilon G(k_n|K)$ we have $\Pi(\omega_\chi) = \omega_{\Pi\circ\chi}$.

Evidently (5) is equivalent to the two conditions:

(a) Φ_S (x) splits completely over K.

(b) The system (6) has a solution compatible with K. In this case the matrix A is determined according to relations (3), and D is determined by the equations $d(g)=\Theta_S \, \nu(g)$ for all $g \varepsilon G$. Moreover, every factorization over K must be equivalent to one of this form.

As remarked at the end of section one, if F is an algebraic number field it is possible to determine the zeros of Φ_S when condition (a) is satisfied. It remains to find a bijection ν and a compatable solution to the system (6). In the remainder of this section we shall consider the problem in detail. A further reduction is possible. Setting $\chi = \Psi = 1$ in (6) we find $\omega_1 = \pm 1$. Since $\{-\omega_\chi\}$ is also a compatible solution, it follows that, in addition to $\Theta_S = \sum d(g)$, we may also assume $\omega_1 = 1$; any factorization is

equivalent to one of this form. Having made this normalization we obtain by setting $\Psi = 1$, the relation

$$(7) \qquad q_S(\nu;\chi) = \left(\frac{p_S(1,\chi)}{\Theta_S}\right)\omega_\chi^{-1} .$$

Inverting this equation gives

$$(8) \qquad \nu(g) = (n\,\Theta_S)^{-1}\sum_\chi \chi(g)\,p_S(1,\chi)\,\omega_\chi^{-1} .$$

From this it follows that ν is uniquely determined by $\{\omega_\chi\}$ and, since the right hand side belongs to K, that K contains a splitting field of Φ_S . Later we will show that there are circumstances in which the minimal extension $K|F$ over which S factors is the splitting field of Φ_S .

If we replace χ by $\chi\Psi$ in equation (7) we have

$$q(\nu;\chi\Psi) = \left(\frac{p_S(1,\chi\Psi)}{\Theta_S}\right)\omega_{\chi\Psi}^{-1} .$$

On the other hand, solving (6) directly for $q(\nu;\chi\Psi)$ and equating the two results gives

$$\omega_\chi\omega_\Psi\omega_{\chi\Psi}^{-1}p_S(1,\chi\Psi) = p_S(\chi,\Psi) \quad \text{for all } \chi,\Psi .$$

Let

$$\omega_\chi\omega_\Psi\omega_{\chi\Psi}^{-1} = u(\chi,\Psi) ;$$

The map $u:\hat{G}\times\hat{G}\to\dot{K}_n$ is a 2-coboundary of $\hat{G}$ with coefficients in $\dot{K}_n$ and has the following properties for all χ and Ψ :

(i) $u(1,1) = 1$

(ii) $u(\chi\,\Psi) = u(\chi,\Psi)$

(iii) $p_S(1,\chi\Psi)u(\chi,\Psi) = p_S(\chi,\Psi)$

(iv) for all $\Pi \in G(K_n|K)$, $\Pi(u(\chi,\Psi)) = u(\Pi o\chi, \Pi o\Psi)$.

We note that

(9) $$\omega_\chi = \prod_{\Psi}^{n} u(\chi ,\Psi) \quad ;$$

whence ω_χ is determined by u up to a factor of a root of unity. In fact, suppose that $\{\omega_\chi'\}$ is any other compatable solution of (6) corresponding to a bijection ν', and such that $\omega_\chi'\omega_\Psi' = u(\chi,\Psi)\omega_{\chi\Psi}'$. Since all ω_χ are non-zero we can write $\omega_\chi' = \varepsilon_\chi \omega_\chi$, where each ε_χ is an n-th root of unity. Then $\varepsilon_\chi\varepsilon_\Psi = \varepsilon_{\chi\Psi}$, so the map $\chi \to \varepsilon_\chi$ is a character of $\hat{G}$; by duality there is an element $e\varepsilon G$ such that $\varepsilon_\chi = \chi(e)$ for all χ. From (8) we have $\nu'(g) = \nu(eg)$; therefore, the corresponding factorization (A', D') is equivalent to (A,D). So if u satisfies the properties (i) - (iv), equation (9) determines, up to equivalence, one factorization of S. It is possible, however, that two different 2 - coboundaries u and u' will satisfy conditions (i)-(iv) unless $p_S(1,\chi)$ is always non-zero (for example, if S is a group matrix then $p_S(1,\chi) = 0$ if $\chi \neq 1$, so only the values of $u(\chi, \chi^{-1})$ are determined by (iii)). Suppose that u and u' both satisfy the conditions and have corresponding factorizations which are equivalent. Then

$\nu'(g) = \nu(eg)$ for some fixed $e \varepsilon G$. Hence $q_S(\nu'; \chi) = \chi(e^{-1})q_S(\nu; \chi)$, so $\{\chi(e^{-1})\omega'_\chi\}$ is a compatible solution of (6). Therefore

$$u' = \omega'_\chi \omega'_\Psi (\omega'_{\chi\psi})^{-1} = u \ .$$

We have shown that there is a one to one correspondence between inequivalent factorizations and 2-coboundaries satisfying conditions (i) through (iv).

Conversely, suppose $K|F$ is an extension which splits $\Phi_S(x)$ and that u is a 2-coboundary with coefficients in $\dot{K}_n$ satisfying (i) through (iv). Let Λ be the Galois algebra determined by u, so that $\Lambda = \bigoplus_{i=1}^{n} K$. Then over Λ we have

$$\begin{aligned} p_S(\chi,\Psi) &= u(\chi,\Psi)\ p_S(1,\chi\Psi) \\ &= (u(\chi,\Psi)w_{\chi\Psi})\ (p_S(1,\chi\Psi)w_{\chi\Psi}^{-1}) \\ &= w_\chi w_\Psi\ q_S(\bar{\nu};\chi\Psi)\ \Theta_S \quad , \end{aligned}$$

where

$$\bar{\nu}(g) = (n\Theta_S)^{-1} \sum_\chi \chi(g)\ p_S(1,\ \chi)w_\chi^{-1} \quad .$$

Since (6) implies (5) if K is replaced with a Galois algebra, we obtain a factorization $S=A^t DA$ over Λ, where $a(1,g) = w_g$ and $d(g) = \Theta_S\ \nu(g)$; whence it is straightforward to show that $s(g,h) = \mathrm{trace}_{\Lambda|K}\ (\bar{\nu}(1)w_g w_h)$. Now using relations (4) this result

implies that there is a group matrix B over K such that $\Phi_C(b(g,h)) = 0$ for all g, h and such that $B^tSB = (\mathrm{trace}_{\Lambda|K}(\bar{\nu}(1)e_g e_h))$ is diagonal. Now, taking $A=B^{-1}$ and $D=B^tSB$ gives a factorization over K. We have proved our main result.

Theorem. Let G be an abelian group of order n, and F a field whose characteristic does not divide 2n. Let S be an nxn F-matrix for which Θ_S is non-zero. Then if $K|F$ is an extension field which splits Φ_S, there is a one to one correspondence between factorizations of S over K and 2-coboundaries u of $\hat{G}$, with coefficients in $\dot{K}_n$ which satisfy conditions (i) through (iv). If C is the structure matrix of a normal basis for the Galois algebra defined by u, then the corresponding equivalence class of factorizations is determined by (A,D) where A is the inverse of some group matrix B satisfying $\Phi_C(b(1,g)) = 0$ for all $g\epsilon G$, and

$$d(g) = n^{-1}\sum_{\chi} \chi(g)\left(\frac{p_S(1,\chi)}{\omega_\chi(A)}\right) . \qquad \text{all } g \in G$$

We remark that in principle this result allows us to compute A and D once u is given (e.g. if $p_S(1,\chi)$ is always non-zero, u is uniquely determined from S) albeit there n! possibilities for B. On the other hand, equations (9) only determine ω_χ up to a factor of an nth root of unity, so there are still many possible solutions $\{\omega_\chi\}$ to test. The situation simplifies when G is an elementary abelian 2-group. In this case $\chi=\chi^{-1}$, and so from (6)

we find

$$\omega_\chi = (-1)^{\varepsilon_\chi} \left(\frac{p_S(\chi,\chi)}{\Theta_S} \right)^{\frac{1}{2}} \quad ; \; \varepsilon_\chi = 0 \text{ or } 1 \quad .$$

Thus if (6) has a solution it must be of this form for some choice of $\{\varepsilon_\chi\}$. Since there are 2^n possibilities there are at most this many inequivalent factorizations.

Because of the large number of possible cases to be checked, we shall give one more necessary condition -- an immediate consequence of the theorem.

Proposition 2. _Assume the hypotheses of the theorem. Then a necessary condition for_ S _to have a factorization over_ K _is that for all_ χ

$$p(\chi,\chi^{-1}) \equiv \Theta_S \pmod{K_\chi} \; ,$$

where $K_\chi = \dot{K}^2$ _when_ $d_\chi = 2$, _and_ K_χ _is the norm group from_ K_d _to_ $K(\zeta_d + \zeta_d^{-1})$ _if_ $d_\chi = d > 2$.

proof. The system (6) implies that

$$\omega_\chi \omega_{\chi^{-1}} = \frac{p_S(\chi,\chi^{-1})}{\Theta_S} \; , \text{ all } \chi \; ,$$

since $q(\nu;1) = 1$ independently of ν. Therefore, as in [2], $\omega_\chi \omega_{\chi^{-1}} \in K_\chi$.

When S is a group matrix, it is easy to show that $p_S(1,\chi) = 0$ for all $\chi \neq 1$; whence the system (6) reduces to the condition stated in proposition two. At the opposite extreme is the case where $p_S(1,\chi)$ is never zero. Then condition (iii) implies that $u(\chi,\Psi) \in \dot{F}_n$ for all χ and Ψ. So in this case u is actually a 2-cocycle of $\hat{G}$ with coefficients in $\dot{F}_n$, and the cohomology class of u in $H^2(\hat{G}; \dot{F}_n)$ will be trivial if and only if S factors over F. Moreover, we have already noted that in this case a factorization, if it exists at all, will be unique up to equivalence. It follows that if $K|F$ is the extension generated by the coefficients of a factorization (A,D) then all factorizations of S generate the same field K. Finally, this uniquely determined extension may be characterized either as the abelian extension of F defined by the 2 - cocycle u as described earlier, or simply as the splitting field of Φ_S. For we already know that K must contain a splitting field L; on the other hand, since $p_S(1,\chi)$ is never zero, we can invert equation (8) and solve for ω_χ. This shows that $\omega_\chi \in L_n$. Therefore $a(1,g) = \sum \chi(g^{-1})\omega_\chi$ also belongs to L, and so $L \supseteq K$.

If n is a prime and F does not contain ζ_n, then every non-group matrix S having a factorization over some extension field of F and such that Θ, is non-zero must satisfy the condition that $p_S(1,\chi)$ is never zero. In order to prove this we will assume $n > 2$. The case $n = 2$ is considered in section three. Now if for some non-trivial $\chi\, p_S(1,\chi) = 0$ then according to (6)

we must also have $q_S\,(\nu;\chi) = 0$. This can be written as

$$\nu\,(1) = -\sum_{\substack{g\varepsilon G \\ g\neq 1}} \chi(g^{-1})\;\nu(g)\;.$$

Since $\nu(1)\varepsilon F$, the right hand side is invariant under all automorphisms in $G(F_n|F)$. But $\{\chi(g)\}_{g\neq 1}$ is an F - basis for F_n, and hence $\nu(g_2) = \ldots. = \nu(g_n)$. Therefore

$$\nu(1) = \quad -\nu(g_2)\sum_{g\neq 1}\chi\;(g^{-1}) = \nu(g_2).$$

This shows that S must be a group matrix.

δ 3. The quadratic case.

This section is devoted to a final example, the case $n = 2$. First suppose $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and $\Theta_S = a+2b+c$ is non-zero. Straightforward computation gives the following

$$p(1,1) = a+b+c$$

$$p(1,\chi) = p(\chi,1) = a-c$$

$$p(\chi,\chi) = a-2b+c\;.$$

So if u is defined by

$$u(1,1) = u(1,\chi) = 1$$

and

$$u(\chi,\chi) = \frac{a-2b+c}{a+2b+c}\quad ,$$

it is easily seen to be a 2-cocycle with coefficients in $\dot{F}$. Hence S is G-congruent (G cyclic of order 2) to a diagonal matrix over the field $K = F(\sqrt{d})$, where

$$d = \frac{a-2b+c}{a+2b+c}$$

The factorization, unique up to equivalence, is given by

$$A = \begin{pmatrix} \frac{1+\sqrt{d}}{2} & \frac{1-\sqrt{d}}{2} \\ \frac{1-\sqrt{d}}{2} & \frac{1+\sqrt{d}}{2} \end{pmatrix} \text{and } D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} ;$$

where

$$d_1 = \tfrac{1}{2}\left(a+2b+c + \frac{a-c}{\sqrt{d}}\right)$$

$$d_2 = \tfrac{1}{2}\left(a+2b+c - \frac{a-c}{\sqrt{d}}\right)$$

It is interesting to note that <u>every</u> S for which Θ_S is non-zero can be diagonalized by a group matrix over some quadratic extension.

Now suppose Θ_S is zero. According to proposition one, if S has a factorization over some R then $S^G = 0$. So we must have

$$a+2b+c=0$$

$$2b = 0$$

$$a + c = 0;$$

Hence S must be of the form $S = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$.

But

$$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = \begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix}$$

holds if and only if the equation

$$a = b(x^2-y^2)$$

is soluable over R. In particular if $R=F$ is a field of characteristic not two then the quadratic form x^2-y^2 is universal; whence any two non-zero matrices of the above form are congruent. If F contains infinitely many elements, then we see that S has infinitely many inequivalent factorizations. This case contrasts sharply with the non-zero case where (for all n) equation (9) implies that there are only finitely many equivalence classes of factorizations.

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